2019-2020 Exam Solutions

SECTION A

1.

E. MIxed isotropic and kinematic hardening

[2 marks]

2.

C. Less conservative than the Goodman line

[2 marks]

3.

A. $u = \frac{3P^2}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + \frac{\pi R^3}{3} + 2\pi R^2\right)$

[2 marks]

SOLUTION 3

Deflection, u, at the position of and in the direction of load, P, is:

$$u = \frac{\partial U}{\partial P}$$

$$\therefore u = \frac{\partial\left(\frac{P^3}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + \frac{\pi R^3}{3} + 2\pi R^2\right)\right)}{\partial P}$$

$$= \boldsymbol{\frac{3P^2}{EI}\left(\frac{L}{4} + \frac{3\pi L^3 R}{12} + \frac{\pi R^3}{3} + 2\pi R^2\right)}$$

4.

A. $\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^2}{2} + M_O\langle x-2\rangle - P\frac{\langle x-4\rangle^2}{2} - q\frac{\langle x-6\rangle^3}{6} + A\right)$

[2 marks]

SOLUTION 4

Integrating $EI\frac{d^2y}{dx^2} = R_A x + M_O\langle x-2\rangle^0 - P\langle x-4\rangle - q\frac{\langle x-6\rangle^2}{2}$ with respect to x gives:

$$EI\frac{dy}{dx} = R_A \frac{x^2}{2} + M_O\langle x-2\rangle - P\frac{\langle x-4\rangle^2}{2} - q\frac{\langle x-6\rangle^3}{6} + A$$

Rearranging this for $\frac{dy}{dx}$:

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^2}{2} + M_O\langle x-2\rangle - P\frac{\langle x-4\rangle^2}{2} - q\frac{\langle x-6\rangle^3}{6} + A\right)}$$

5.

A. $\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^3}{6} + \frac{M_O\langle x-2\rangle^2}{2} - P\frac{\langle x-4\rangle^3}{6} - q\frac{\langle x-6\rangle^4}{24} + Ax + B\right)$

[2 marks]

SOLUTION 5

Integrating $\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^2}{2} + M_O\langle x-2\rangle - P\frac{\langle x-4\rangle^2}{2} - q\frac{\langle x-6\rangle^3}{6} + A\right)$ with respect to x gives:

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{EI}\left(R_A \frac{x^3}{6} + \frac{M_O\langle x-2\rangle^2}{2} - P\frac{\langle x-4\rangle^3}{6} - q\frac{\langle x-6\rangle^4}{24} + Ax + B\right)}$$

6.

A. True

[2 marks]

7.

A. A

[2 marks]

SOLUTION 7

Point A is the **furthest point from the neutral axis and on the opposite side** to point E.

8.

B. $EI\frac{d^2y}{dx^2} = M$

[2 marks]

9.

D. The deflection in the perpendicular direction to the loading direction is always zero.

[2 marks]

10.

D. Reduces the load required to cause buckling

[2 marks]

SOLUTION 10

All buckling load equations have length, L, on the bottom of the equation. Therefore, increasing the length of the beam reduces the load required to cause buckling. An example, for a hinged-hinged beam, is shown below.

$$P_c = \frac{\pi^2 EI}{L^2}$$

11.

C. 48.0 MPa

[2 marks]

SOLUTION 11

Given $\sigma_z = 125$ MPa, $\sigma_z = 50$ MPa and $\tau_{zy} = 30$ MPa:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{z\theta}^2} = \sqrt{\left(\frac{125 - 50}{2}\right)^2 + 30^2} = \mathbf{48.0\ MPa}$$

12. CJB Qu

B. 74 °C

[2 marks]

SOLUTION 12

$$\Delta T_{min} = \frac{\delta d}{d_0 \alpha} = \frac{0.03}{24 \times 17 \times 10^{-6}} = \mathbf{73.5\ °C}$$

13. CJB Qu

E. 312 MN/m

[2 marks]

SOLUTION 13

The axial stiffness for a 1D truss element is given by:

$$k = \frac{AE}{L} = \frac{0.015 \times 0.03 \times 208 \times 10^9}{0.3} = \mathbf{312 \times 10^6\ N/m}$$

14. CJB Qu

D. 60 MPa

[2 marks]

SOLUTION 14

The maximum shear stress at the N.A. given by:

$$\tau = \frac{SA\bar{y}}{Iz} = \frac{12 \times 40000 \times (25 \times 20) \times 10}{25 \times 40^3 \times 25} = \mathbf{60\ MPa}$$

15. CJB Qu

E. 26 MPa

[2 marks]

SOLUTION 15

$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

at r = 0.025 (ID), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.025^2} - \frac{3+0.3}{8} 8000 \times 314.15^2 \times 0.025^2$$

$$0 = A - 1600B - 2.04 \times 10^5$$

at r = 0.2 (OD), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.2^2} - \frac{3+0.3}{8} 8000 \times 314.15^2 \times 0.2^2$$

$$0 = A - 25B - 13.0 \times 10^6$$

$$0 = -1575B + 12.8 \times 10^6$$

therefore:

$$B = 8142$$

$$A = 13.2 \times 10^6$$

Hoop stress at the bore is given by:

$$\sigma_\theta = 13.2 \times 10^6 + \frac{8142}{0.025^2} - \frac{1+3\nu}{8} 8000 \times 315.14^2 \times 0.025^2 = \mathbf{26.3\ MPa} \approx \mathbf{26\ MPa}$$

16. CJB Qu

D. 209 MPa

[2 marks]

SOLUTION 16

Torsional shear stress:

$$\tau = \frac{Tr}{J} = \frac{32 \times 500 \times 12.5 \times \ 10^{-3}}{\pi \times (25 \times 10^{-3})^4} = 163 \text{ MPa}$$

Axial stress:

$$\sigma_a = \frac{F}{A} = \frac{40000}{\pi \times (12.5 \times 10^{-3})^2} = 81.5 \text{ MPa}$$

Centre given by:

$$C = \frac{\sigma_a}{2} = \frac{81.5}{2} = 40.75 \text{ MPa}$$

Radius is given by:

$$R = \sqrt{\left(\frac{\sigma_a}{2}\right)^2 + \tau^2} = \sqrt{40.75^2 + 163^2} = 168 \text{ MPa}$$

Max principal stress given by:

$$\sigma_1 = C + R = 40.75 + 168 = \mathbf{208.75} \ \approx \mathbf{209\ MPa}$$

17. CJB Qu

A. More conservative

[2 marks]

18. CJB Qu

D. Deviatoric Plane

[2 marks]

19. CJB Qu

A. $$[k_{AB}] = \left(\frac{AE}{L}\right)\begin{bmatrix} 0.75 & -\sqrt{3}/4 & -0.75 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 0.25 & \sqrt{3}/4 & -0.25 \\ -0.75 & \sqrt{3}/4 & 0.75 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -0.25 & -\sqrt{3}/4 & 0.25 \end{bmatrix}$$

[2 marks]

20. CJB Qu

C. 43 mm

[2 marks]

SOLUTION 20

According to the von Mises yield criterion:

$$\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

For a shaft under pure torque, the Mohr's circle is centred on the origin and σ_1 and σ_2 will be the same magnitude and will also be the maximum allowable shear stress, therefore:

$$\sigma_y^2 = 3R^2$$

or

$$250^2 = 3R^2$$

giving:

$$R = \sqrt{\frac{250^2}{3}} = 144.34\ \text{MPa}$$

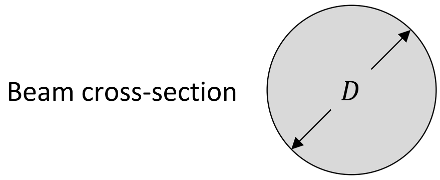
$$144.34 \times 10^6 = \frac{Tr}{J} = \frac{2T}{\pi r^3}$$

$$r = \sqrt[3]{\frac{2 \times 18000}{144 \times 10^6 \pi}} = \mathbf{0.043 = 43\ mm}$$

SECTION B

21.

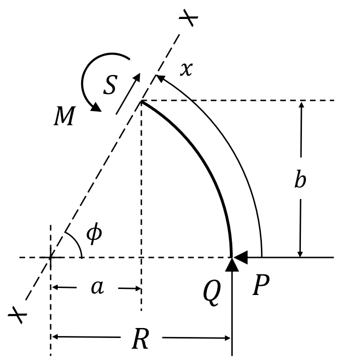
Second Moment of Area, I, calculation:



$$\therefore I = \frac{\pi D^4}{64}$$

[2 marks]

Free Body Diagram:



[3 marks]

Taking moments about X-X:

$$M + Q(R - a) = Pb$$

$$\therefore M = PRsin\phi - Q(R - Rcos\phi) = PRsin\phi - QR(1 - cos\phi)$$

[3 marks]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U = \int \frac{M^2}{2EI} dx = \frac{R}{2EI} \int_0^{\pi} \left(PRsin\phi - QR(1 - cos\phi)\right)^2 d\phi$$

where,

$$ds = Rd\phi$$

[3 marks]

Vertical displacement:

$$u_v = \frac{\partial U}{\partial Q} = \frac{R}{EI}\int_0^{\pi}\left(PRsin\phi - QR(1-cos\phi)\right)\left(-R(1-cos\phi)\right)d\phi$$

[3 marks]

Setting dummy load, $Q = 0$,

$$u_v = \frac{PR^3}{EI}\int_0^{\pi} sin\phi(cos\phi - 1)d\phi$$

$$= \frac{PR^3}{EI}\int_0^{\pi}(sin\phi cos\phi - sin\phi)d\phi$$

[1 mark]

Trig identity,

$$sin2\phi = 2sin\phi cos\phi$$

$$\therefore sin\phi cos\phi = \frac{1}{2}sin2\phi$$

Substituting this into (i),

$$u_v = \frac{PR^3}{EI}\int_0^{\pi}\left(\frac{1}{2}sin2\phi - sin\phi\right)d\phi = \frac{PR^3}{EI}\left[-\frac{1}{4}cos2\phi + cos\phi\right]_0^{\pi}$$

$$\therefore u_v = -\frac{2PR^3}{EI}$$

[2 marks]

Substituting in the expression for second moment of area, I,

$$\therefore u_v = -\frac{128PR^3}{E\pi D^4}$$

[1 mark]

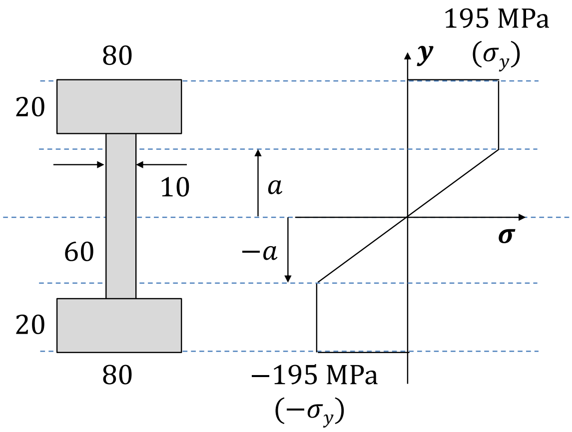
I.e. downwards deflection of $\boldsymbol{\frac{128PR^3}{E\pi D^4}}$. This is therefore the required expression for the minimum step height, h.

[2 marks]

22.

(a)

Assuming yielding occurs at $y \geq a$ and $y \leq -a$, the stress distribution in the beam cross-section is as follows:



Variation of stress with y:

- For $a < y < 50$, $\sigma = 195$ MPa
- For $-a < y < a$, $\sigma = \frac{195}{a} y$ MPa
- For $-50 < y < -a$, $\sigma = -195$ MPa

[3 marks]

<u>Moment equilibrium</u>

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment)

$$M = \int_A y\sigma dA = \int_y y\sigma b dy$$

[1 mark]

Due to the symmetry of the stress distribution above and below the Y-Y axis, and substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left\{\int_0^a y\frac{\sigma_y}{a}yb_w dy + \int_a^{30} y\sigma_y b_w dy + \int_{30}^{50} y\sigma_y b_f dy\right\}$$

[2 marks]

$$\therefore \frac{M}{2\sigma_y} = \frac{b_w}{a}\int_0^a y^2 dy + b_w\int_a^{30} y dy + b_f\int_{30}^{50} y dy$$

$$= \frac{b_w}{a}\left[\frac{y^3}{3}\right]_0^a + b_w\left[\frac{y^2}{2}\right]_a^{30} + b_f\left[\frac{y^2}{2}\right]_{30}^{50}$$

$$= \frac{b_w}{3a}(a^3) + \frac{b_w}{2}(30^2 - a^2) + \frac{b_f}{2}(50^2 - 30^2)$$

$$= -\frac{b_w}{6}a^2 + 450b_w + 800b_f$$

$$\therefore \frac{b_w}{6}a^2 = 450b_w + 800b_f - \frac{M}{2\sigma_y}$$

[2 marks]

$$\therefore a = \sqrt{2700 + 4800\frac{b_f}{b_w} - \frac{3M}{\sigma_y b_w}}$$

[2 marks]

$$= \mathbf{19.8\ mm}$$

[2 marks]

(b)

Assuming unloading to be entirely elastic. Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta\sigma}{y}$$

[1 mark]

Where,

$$I = \frac{bd^3}{12} - 2\left(\frac{b_i {d_i}^3}{12}\right) = \frac{80 \times 100^3}{12} - 2\left(\frac{35 \times 60^3}{12}\right) = 5{,}406{,}666.67\ \mathrm{mm}^2$$

[1 mark]

Max change in stress ($\Delta\sigma$) will occur at $y = \pm 50$ mm (furthest perpendicular distance from the Y-Y axis):

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-26{,}460{,}000 \times \pm 50}{5{,}406{,}666.67}$$

$$= \mp 244.7 \text{ MPa}$$

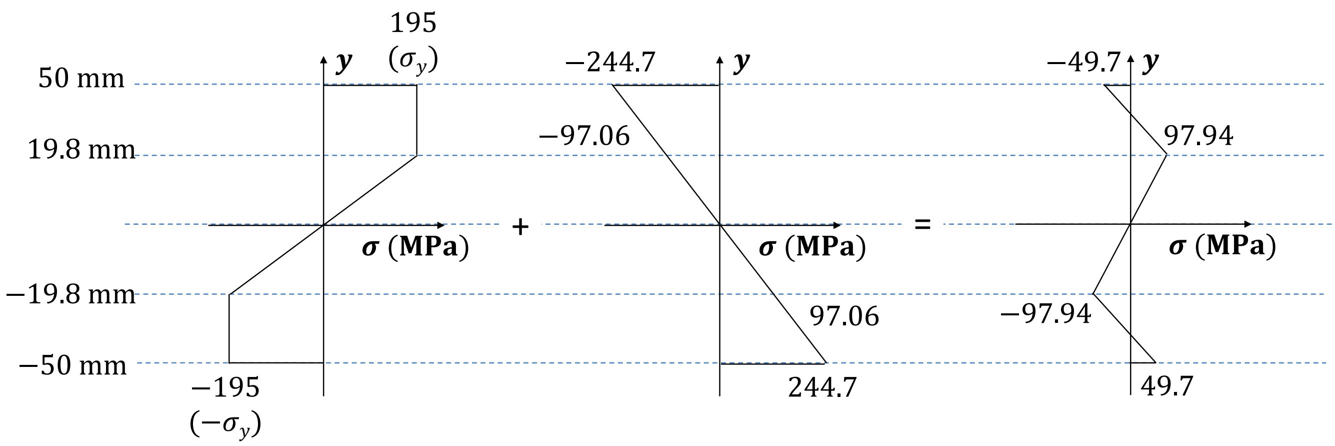
i.e.:

$$\text{at } y = 50 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = -244.7 \text{ MPa}$$

$$\text{and at } y = -50 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = 244.7 \text{ MPa}$$

[2 marks]

Therefore, the stress distribution after unloading can be calculated as:



Loading stress distribution

(Elastic) unloading stress distribution

Residual stress distribution

Interpolation of (elastic) unloading line:

At $y = 50$ mm, $\sigma = -244.7$ MPa

$$y = m\sigma + c$$

$$\therefore 50 = \text{m} \times -244.7 + 0$$

$$\therefore \text{m} = -0.204$$

At $y = 19.8$ mm, $19.8 = -0.204 \times \sigma$

$$\therefore \sigma = -97.06 \text{ MPa}$$

[3 marks]

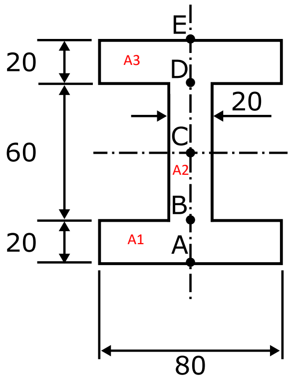
The largest residual stress magnitude is ±97.94 MPa, so reverse yielding does not occur, as this is below yield of the materials, and so the assumption of purely elastic unloading is reasonable.

[1 mark]

23.

(a)

Calculate the second moment of area of the section about the neutral axis (N.A.)



All dimensions in mm

$$I = \sum\left(\frac{BD^3}{12} + A(\bar{y}_n - \bar{y})^2\right)$$

[1 mark]

	W	B	$\bar{y}_n$	A	$\overline{Ay_n}$	$\frac{BD^3}{12}$	
A1	80	20	-40	1600	2560000	53333	2.61E+06
A2	20	60	0	1200	0	360000	3.60E+05
A3	80	20	40	1600	2560000	53333	2.61E+06
						I_tot	5.59E+06

[4 marks]

$$\tau = \frac{SQ}{Iz}$$

[1 mark]

A & E are free surfaces, so:

$$\tau_A = \tau_E = 0$$

[1 mark]

Can use area below at B, 2 values due to section change:

$$\tau_{B1} = \frac{32000 \times (20 \times 80) \times (40)}{5.59 \times 10^6 \times 80} = \mathbf{4.6\ MPa}$$

[2 marks]

$$\tau_{B2} = \frac{32000 \times (20 \times 80) \times (40)}{5.59 \times 10^6 \times 20} = \mathbf{18.3\ MPa}$$

[2 marks]

At C (centroid), need to consider the effect of two areas:

In this case $Q = \sum A(y_n - y)$

$$Q = \big((20 \times 80) \times (40)\big) + \big(20 \times 30 \times (15)\big)$$

$$Q = 64000 + 9000 = 73000$$

and therefore:

$$\tau_G = \frac{32000 \times 73000}{5.59 \times 10^6 \times 20} = \mathbf{20.9MPa}$$

[3 marks]

At D, 2 values, which are the same as B (symmetry)

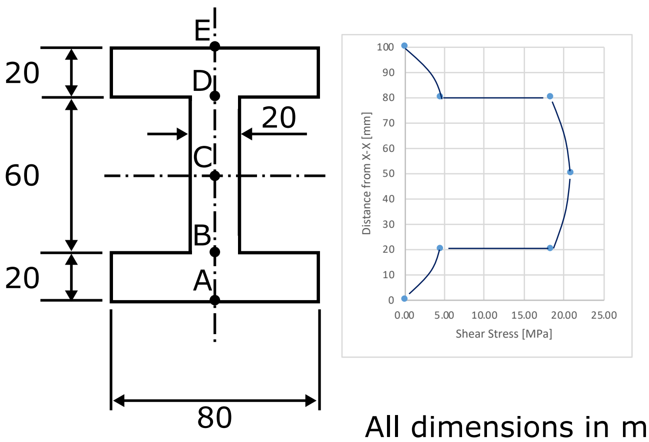
$$\tau_{D1} = \frac{32000 \times (20 \times 80) \times (40)}{5.59 \times 10^6 \times 20} = \mathbf{18.3\ MPa}$$

[1 mark]

$$\tau_{D2} = \frac{32000 \times (20 \times 80) \times (40)}{5.59 \times 10^6 \times 80} = \mathbf{4.6\ MPa}$$

[1 mark]

(b)



All dimensions in mm

[4 marks]